UDC 519.248.6

Some Characteristic Properties of Resource Distribution Nonuniformity in Economic Systems

Leonid A. Sevastianov*, **Alexander V. Kryanev**†, **Daria E. Sliva**†, **Valentin V. Matokhin**‡

* *Department of Applied Probability and Informatics,*
Peoples' Friendship University of Russia (RUDN University),
6 Miklukho-Maklaya str., Moscow, 117198, Russia
† *National Research Nuclear University "MEPhI"*
31 Kashirskoe shosse, Moscow, 115409, Russian Federation
‡ *ZAO "TEKORA"*
65 Profsoyuznaya st., Moscow, 117997, Russian Federation

Email: sevastianov_la@rudn.university, avkryanev@mephi.ru, desliva@mephi.ru,vmatokhin@tekora.ru

In this article the research on two-parameter Lorenz curve approximants is continued. Several characteristic properties of one such family of two-parameter functions used to approximate the Lorenz curve is studied. Several interpretations of the functional parameters are introduced. Distribution density functions that correspond to various values of parameters of the class of functions used to approximate the Lorenz curve are analysed. The implications and the value brought to the study of nonuniformity of resource distributions by this introduction is discussed. The ties between the values of the parameters and statistic estimators are tested. It is concluded that the values of the approximant parameters are connected to the coefficient of kurtosis and the value of the interquartile range. An interpretation of the approximant parameters with regards to economic inequality is suggested. The special case corresponding to uniform distribution of the shares of the resource allocated to separate agents is distinguished and discussed. It is concluded that it is of special value with regards to solving the problem of finding the optimal distribution of a resource within an economic system.

Key words and phrases: resource distribution, Lorenz curve, economic inequality.

In: K. E. Samouylov, L. A. Sevastianov, D. S. Kulyabov (eds.): Selected Papers of the VIII Conference "Information and Telecommunication Technologies and Mathematical Modeling of High-Tech Systems", Moscow, Russia, 20-Apr-2018, published at http://ceur-ws.org

1. Introduction

One possible representation of a empiric distribution of resources within an economic system is a Lorenz curve. Various single parameter and multiple parameter approximants of the Lorenz curve, the corresponding probability distribution functions and the probability density functions have been studied extensively [1–8]. As the Lorenz curve is currently the predominant apparatus used to characterise and simulate inequality in economic and social systems, both its analysis and applications are of great interest. The present paper constitutes an inquiry into one of its many proposed approximants and its properties. This is a follow-up research from previous elaboration on the class of two-parameter functions [3, 4].

2. Main section

Let G be a resource distributed among N economic agents. Let $i = 1, \ldots, n, \ldots, N$ be the indices of G_i — the amount of resource G allocated to the,

$$\sum_{i=1}^{N} G_i = G.$$

A resource distribution G_i, $i = 1, \ldots, N$ may be represented by Lorenz curve [7, 8]. For that purpose the original series is ranged in ascending order: $G_1 \leqslant G_2 \leqslant \ldots \leqslant G_N$.

Next the accumulated sums

$$S_n = \sum_{i=1}^{n} G_i$$

are calculated and plotted on a plane with axes $x = n/N$ and $y = S_n/S_N$.

Previously [3,4] it has been demonstrated that the Lorenz curves may be approximated my the two-parameter curve belonging to the class of functions

$$y(x, \alpha, \beta) = 1 - (1 - x^{\alpha})^{1/\beta},$$

where $1 \leqslant \alpha < \infty$, $1 \leqslant \beta < \infty$.

It has been previously demonstrated [5, 6] that parameters α and β may serve as indicators of resource distribution non-uniformity. At $\alpha = \beta = 1$ the accumulated sums $y(x, \alpha, \beta) = x$ and the corresponding resource distribution is absolutely uniform, which means that each economic agent acquires possesses exactly G/N of the total accumulated wealth. In the opposite scenario, while $\alpha \to \infty$, $\beta \to \infty$ the resource distribution approaches an extremely non-uniform distribution, where the agent of the highest rank has exclusive control over the entire stock of wealth and the agents of lower rank have nothing.

By manipulating the values of the parameters α and β various distribution density functions may be attained. It has previously been shown [7, 8] that even the single parameter class of curves given by $y(x, \alpha, \beta) = 1 - (1 - x^{\alpha})^{1/\alpha}$, demonstrate a good fit for empiric wealth distributions. However, the implementation of the two-parameter class of functions 2 enables to improve the quality of the fit by allowing for asymmetric resource distributions, while the class of functions **??** can only simulate a symmetric (with regards to the main diagonal of the Lorenz square) distribution [9, 10]. Besides, the use of the two-parameter approximant allows to more closely fit the intervals that correspond to groups of the highest and the lowest rank, while sustaining the ease of interpretation and calculation that were part of the incentive for the use of the single parameter approximant [11–14].

Fig. 1–2 illustrate the correspondence between selected values of the parameters α and β and the corresponding frequency histograms. In this paper the discrete values case is considered, for the continuous case and the analytical results see [3, 4]. Here we scrutinise the distribution of the total resource $G = 1000$ among $N = 1000$ economic agents with regards to variation of parameters α and β. In fig. 1-2 the relative shares of the resource G are plotted along the horizontal axis and the relative number of agents that possess that portion of the shared resource are plotted along the vertical axis. The histograms are standardised so that the 1000 values of $G_i, i = 1, \ldots, N$ are always grouped into 20 equal-sized intervals.

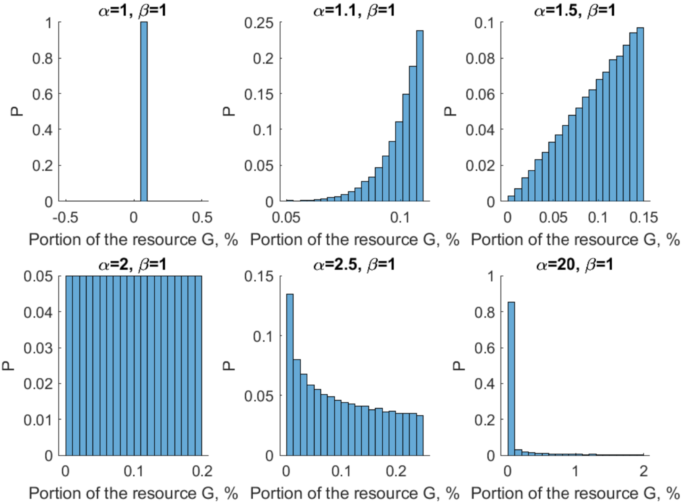

Figure 1. The resource distribution histograms for various values of α ($\beta = 1$)

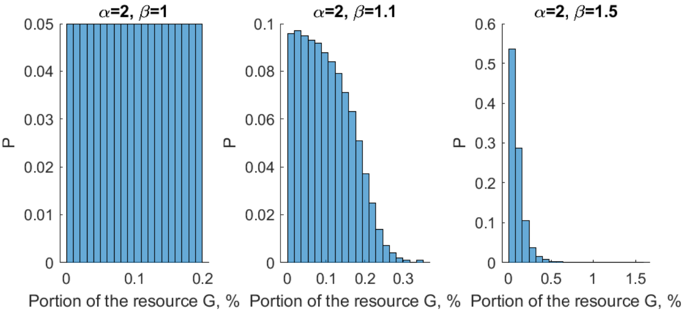

Figure 2. The resource distribution histograms for various values of β ($\alpha = 2$)

It is evident that for $\alpha = 1, \beta = 1$ all agents acquire an equal share of the resource $\overline{G}$ [15]. While α increases while the value of β is fixed, the uniformity of the distribution is compromised as the portion of the total resource allocated to single agents are varied. Provided that every percentile of the resource distribution in the context of societal wealth represents a class, at $\alpha \to 2, \beta = 1$, probability that any given agent G_i is positioned within any given class is uniform. This does not mean that the shares G_i are equal. In fact, when $\alpha = 2, \beta = 1$, G_i varies from 0 to $2 \cdot \overline{G}$.

It may be noted that the distribution that corresponds to the Lorenz curve

$$y(x, \alpha, \beta) = 1 - (1 - x^{\alpha}), \quad \alpha = 2$$

is of special interest for two reasons:

– the maximal share G_i that may be allocated to an agent or a group of agents is limited by $G_i = 2 \cdot \overline{G}$;
– the probability that an agent will be positioned within any percentile, decile, quartile is equal.

The density function of the considered distribution with respect to G is as follows

$$f(G_i) = \begin{cases} \dfrac{1}{2 \cdot \overline{G}}, & x \in [0, 2 \cdot \overline{G}], \\ 0 : & x \notin [0, 2 \cdot \overline{G}]. \end{cases}$$

The Gini coefficient K_G — an extensively implemented measure of distribution non-uniformity [16–21] — for the curve given by 2 is equal to $1 - \frac{2}{\alpha+1}$, or, for the case in consideration $K_G = \frac{1}{3}$. It must be noted that high ($K_G > 0.5$) values of K_G are rarely observed with regard to societal economic inequality and at present time only occur in undeveloped and some developing countries.

It is further proposed that the values of parameters α and β of the class of functions 2 specify distinct statistical properties of empiric distributions of total accumulated wealth within a society among its members.

The parameter α is a characteristic of peakedness of the probability density function. The resource density function will become less flat and more pointed as the value of parameter α increases. For lower values of α the values of G_i are concentrated closer to the mean value

$$\overline{G} = \sum_{i=1}^{N} G_i = G/N,$$

while for higher values of α the values of G_i are scattered further from $\overline{G}$. This means that as the value of α increases, less and less agents are allocated the 'fair' portion of the resource $\overline{G}$. As to the statistical properties, the leptokurtic probability density functions are classified by a higher α coefficient of kurtosis, calculated as

$$\gamma_2 = \frac{\mu_4}{\sigma^4} - 3,$$

where $\mu_4 = \mathbf{E}[(G - \mathbf{E}G)^4]$ is the kurtosis of the stochastic value G, and σ is its standard deviation.

On the other hand, the parameter β is characteristic of the disparity between the proportion of the resource belonging to the agents of the lowest and the highest rank. From an economic point of view the increase in the value of β is accompanied by societal polarity which is characterised by the formation of two segregated groups of ultra-rich and ultra-poor. The value of the interquartile range (IQR) is determined primarily by the value of β. IQR is the difference between the median value of the resource allocated to the 50% of the highest ranking agents and the median value of the resource allocated to 50% lowest ranking agents. With regards to measuring economic inequality the interquartile range is a measure of the gap between the median wealth of the 50% of the richest and the median wealth of the 50% of the poorest. The value of the interquartile range is a robust measure of variation in comparison with integral range, a characteristic of spread of a stochastic value.

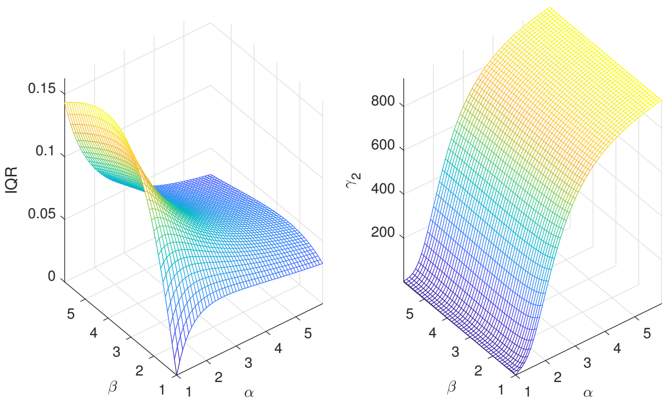

Figure 3. Plots of kurtosis and IQR for various values of β and $\alpha = 2$

Fig. 3 illustrates the correspondence between the values of the interquartile range and the coefficient of kurtosis with regards to various values of parameters α and β. The values of the parameters α and β are plotted along the axes of abscissa and ordinates, while the values of the measures of distribution in question are plotted along the applicate axis.

It is clear that the value of the coefficient of kurtosis of the considered class of distributions is chiefly determined by the value of the parameter α. It can be concluded that the parameter β, in comparison to the weight of the parameter α, scarcely affects the value of the coefficient of kurtosis of the distribution density and is largely insignificant with regards to measuring the flatness of the distribution. It is also shown that the maximal value of the interquartile range is attained at large values of the parameter β while $\alpha = 1$. Therefore we propose that the parameter α may be interpreted as a characteristic of the gap between the wealth of those of the highest rank and those of the lowest rank, while the parameter β is a measure of variation.

3. Conclusions

The attained results are useful for solving various problems associated with finding optimal resource allocation among several objects or agents within an economic system. The proposed interpretation of the parameters α and β of the two-parameter Lorenz curve approximant makes it possible to subsequently define classes and clusters of empiric economic resource distributions and the corresponding economic and and societal distributions. The question whether there exists an empiric relationship between the resource distribution that corresponds to the special case of the studied Lorenz curve approximant and the performance of real economic systems is a subject of further investigation.

References

1. M. O. Lorenz, Methods of Measuring the Concentration of Wealth, Publications of the American Statistical Association **9** (70) (1905) 209–219.
2. Nanak C. Kakwani, Applications of Lorenz curves in economic analysis. Econometrica: Journal of the Econometric Society **45** (3) (1977) 719–727.
3. I. Antoniou, V. V. Ivanov, Yu. L. Korolev, A. V. Kryanev, V. V. Matokhin, Z. Suchanecki, Analysis of resources distribution in economics based on entropy. Physica A, no. 304 (2002) 525–534.
4. I. Antoniou, V. V. Ivanov, A. V. Kryanev, V. V. Matokhin, M. V. Shapovalov, On the efficient resources distribution in economics based on entropy, in: Physica A, no. 336 (2004) 549–562.
5. A. V. Kryanev, V. V. Matokhin and V. V. Haritonov, Entropic method of controlling production of an economic system, Economic strategies, no. 5 (2010), in Russian.
6. A. V. Kryanev, V. V. Matokhin and S. G. Klimanov, Statistical functions of resource distribution in economics, Preprint MEPhI, 010-98, Moscow, 1998, in Russian.
7. C. Kleiber, S. Kotz. Statistical size distributions in economics and actuarial sciences, John Wiley & Sons, 2003.
8. S. K. Singh, and Gary S. Maddala, A function for size distribution of incomes, Modeling income distributions and Lorenz curves. Springer, New York (2008) 27–35.
9. Daniel Slottje, Modeling income distributions and Lorenz curves, Journal of Economic Inequality, **8** (4) (2010) 525–526.
10. M. Ibragimov, R. Ibragimov, P. Kattuman, et al. Econ Theory (2017). `https://doi.org/10.1007/s00199-017-1037-0`
11. José María Sarabia, Vanesa Jordá, and Carmen Trueba, The Lamé class of Lorenz curves, Communications in Statistics-Theory and Methods **46** (11) (2017) 5311–5326.
12. Miguel A. Sordo, Jorge Navarro, and José María Sarabia, Distorted Lorenz curves: models and comparisons, Social Choice and Welfare **42** (4) (2014) 761–780.
13. Melanie Krause, Parametric Lorenz curves and the modality of the income density function, Review of Income and Wealth **60** (4) (2014) 905–929.
14. E. Gómez-Déniz, A family of arctan Lorenz curves. Empirical Economics **51** (3) (2016) 1215–1233.
15. Thomas Kämpke, and Franz Josef Radermacher, Lorenz Densities and Lorenz Curves, Income Modeling and Balancing. Springer, Cham, 2015, 29–54.
16. Rolf Aaberge, Characterizations of Lorenz curves and income distributions, Social Choice and Welfare **17** (4) (2000): 639–653.
17. P. Ortega, et al., A new functional form for estimating Lorenz curves, Review of Income and Wealth **37** (4) (1991) 447–452.
18. J-M. Sarabia, Enrique Castillo, and Daniel J. Slottje, An ordered family of Lorenz curves, Journal of Econometrics **91** (1) (1999) 43–60.
19. James Davies, and Michael Hoy, Making inequality comparisons when Lorenz curves intersect, The American Economic Review **85** (4) (1995) 980–986.
20. Claudio Zoli, Intersecting generalized Lorenz curves and the Gini index, Social Choice and Welfare **16** (2) (1999) 183–196.
21. Barry C. Arnold, The Lorenz curve: evergreen after 100 years, Advances on Income Inequality and Concentration Measures. Routledge, 2001. 34-46.